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Approaching pooling design with smaller efficient ratio

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Abstract In this paper, we employ affine symplectic space $ASG(2\nu, \mathbb{F}_q)$ as a tool to construct two new classes of d^e-disjunct matrices. The efficiency ratio of new d^e-disjunct matrices is smaller than that of D'yachkov et al. (J Comput Biol 12:1129–1136, 2005).

Keywords Pooling designs · s^e-Disjunct matrix · Affine symplectic space

1 Introduction

Given n items with at most d positive ones, we study the problem of identifying all positive items with the minimum number of tests. Each test is on a subset of items, called a *pool*. The

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test-outcome is *positive* if the pool contains at least one positive item and otherwise, the outcome is *negative*. This problem is called *group testing*. Group testing has many applications [1,2,14].

For some applications of group testing, especially in molecular biology [2,7,9,10], pools are required to be designed at the beginning, that is, no test-outcome of a pool can effect design of another pool. Such a group testing is called *nonadaptive group testing*. Nonadaptive group testing can be described by a 0–1 matrix M with items as column indices and pools as row indices; the entry in cell (p, i) is 1 if and only if item i is in pool p.

For each possible set P of positive items, test-outcomes on all pools form a 0–1 column vector, in which each entry is 1 if and only if corresponding row index (pool) p contains a positive item, that is, columns with indices in P have an entry 1 at row p. Therefore, if we consider each column 0–1 vector as a characteristic vector of a subset of pools, then test-outcome vector is the characteristic vector of union of pool subsets characterized by columns with indices in P. For simplicity of statement, we will say that *test-outcome vector is the union of columns with indices in* P.

If M can identify all positive items, then M has to give different test-outcomes for different possible sets of positive items. Therefore, M can identify all positive items, under assumption that there are at most d positive items, if and only if all unions of at most d columns are distinct. Such a 0–1 matrix M is called a $\bar{d}$-separable matrix.

Since there is no efficient way to decode the set of all positive items from test-outcome of a $\bar{d}$-separable matrix, one usually uses a d-disjunct matrix, instead of $\bar{d}$-separable matrix, to do the job. A 0–1 matrix M is *d-disjunct* if no column can be contained in the union of d other columns. Every d-disjunct matrix is $\bar{d}$-separable. However, the inverse is not true. To decode a test-outcome, we need only to delete all items in negative pools and then remainders are all positive.

A 0–1 matrix M is *d^e-disjunct* if given any $d + 1$ columns of M with one designated, there are e rows with a 1 in the designated column and 0 in each of the other d columns [8]. An d^1-disjunct matrix is the same as a *d-disjunct* matrix. An d^e-disjunct matrix is *fully d^e-disjunct* if it is not $s^{e'}$-disjunct whenever $s > d$ or $e' > e$ [3]. The d^e-disjunct matrix and the fully d^e-disjunct matrix are used for error-tolerant testing. They can detect errors from test-outcome under certain condition.

All above mentioned 0–1 matrices M for nonadaptive group testing are called *pooling designs*. There are many ways to do pooling design. The *containment design* is one of them. It was initiated by Macula [8] and many researchers have made efforts to enrich the containment design [3,4,6,9,11–13,16]. It gives the small ratio between the number t of pools and the number n of items [4]. This ratio is called the *efficiency ratio*.

In this paper, we employ affine symplectic space $ASG(2\nu, \mathbb{F}_q)$ to construct two new classes of d^e-disjunct matrices. We will also compare the efficiency ratio of new designs with that of [3].

2 Affine symplectic space

Let

$$K = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}.$$

The *symplectic group* of degree 2ν over $\mathbb{F}_q$, denoted by $Sp_{2\nu}(\mathbb{F}_q)$, consists of all $2\nu \times 2\nu$ matrix T over $\mathbb{F}_q$ satisfying $TKT^t = K$. There is an action of $Sp_{2\nu}(\mathbb{F}_q)$ on $\mathbb{F}_q^{(2\nu)}$ defined by

$$\begin{aligned} \mathbb{F}_q^{(2\nu)} \times Sp_{2\nu}(\mathbb{F}_q) &\longrightarrow \mathbb{F}_q^{(2\nu)} \\ ((x_1, x_2, \ldots, x_{2\nu}), T) &\longmapsto (x_1, x_2, \ldots, x_{2\nu})T. \end{aligned}$$

The vector space $\mathbb{F}_q^{(2\nu)}$ together with the above group action of the symplectic group $Sp_{2\nu}(\mathbb{F}_q)$, is called the *2ν-dimensional symplectic space over* $\mathbb{F}_q$ [15]. An m-dimensional subspace P is said to be of type (m, r), if PKP^t is of rank $2r$. In particular, subspaces of type $(m, 0)$ are called m-dimensional *totally isotropic subspaces.*The subspaces of type (m, r) exist if and only if $2r \le m \le \nu + r$. The subspace of type (m, r), which contains subspaces of type (m_1, r), exists if and only if $2r \le m_1 \le m \le \nu + r$. From [15], the number of subspaces of type (m, r), denoted by $N(m, r; 2\nu)$, is given by

$$N(m, r; 2\nu) = \frac{\prod_{i=\nu+r-m+1}^{\nu}(q^{2i}-1)}{\prod_{i=1}^{r}(q^{2i}-1)\prod_{i=1}^{m-2r}(q^i-1)}. \tag{1}$$

Let $N(m_1, r; m, r; 2\nu)$ denote the number of subspaces of type (m_1, r) contained in a given subspace of type (m, r). From [15],

$$N(m_1, r; m, r; 2\nu) = q^{2r(m-m_1)}\frac{\prod_{i=m-m_1+1}^{m-2r}(q^i-1)}{\prod_{i=1}^{m_1-2r}(q^i-1)}. \tag{2}$$

Let $N'(m_1, r; m, r; 2\nu)$ denote the number of subspaces of type (m, r) containing a given subspace of type (m_1, r). From [15],

$$N'(m_1, r; m, r; 2\nu) = \frac{\prod_{i=1}^{\nu+r-m_1}(q^{2i}-1)}{\prod_{i=1}^{\nu+r-m}(q^{2i}-1)\prod_{i=1}^{m-m_1}(q^i-1)}. \tag{3}$$

Suppose P is a subspace of type (m, r) in 2ν dimension symplectic space $\mathbb{F}_q^{(2\nu)}$. A coset of $\mathbb{F}_q^{(2\nu)}$ relative to a subspace P of type (m, r) is called an *(m, r)-flat*. A flat F_1 is said to be *incident* with a flat F_2, if F_1 contains or is contained in F_2. The point set $\mathbb{F}_q^{(2\nu)}$ with all the flats and the incidence relation among them defined above is said to be the 2ν-dimensional *affine symplectic* space, denoted by $ASG(2\nu, \mathbb{F}_q)$.

The set of matrices of the form

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}$$

where $T \in Sp_{2\nu}(\mathbb{F}_q)$ and $v \in \mathbb{F}_q^{(2\nu)}$, forms a group under matrix multiplication. This group is said to be the *affine symplectic* group of $ASG(2\nu, \mathbb{F}_q)$, denoted by $ASp_{2\nu}(\mathbb{F}_q)$. Define the action of $ASp_{2\nu}(\mathbb{F}_q)$ on the $ASG(2\nu, \mathbb{F}_q)$ as follows:

$$\begin{aligned} ASG(2\nu, \mathbb{F}_q) \times ASp_{2\nu}(\mathbb{F}_q) &\to ASG(2\nu, \mathbb{F}_q) \\ \left(x, \begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}\right) &\mapsto xT + v. \end{aligned}$$

Then affine-symplectic group $ASp_{2\nu}(\mathbb{F}_q)$ acts transitive on the set of (m, r)-flats in $ASG(2\nu, \mathbb{F}_q)$ [15].

Lemma 2.1 *Let $ASG(2\nu, \mathbb{F}_q)$ denote the 2ν-dimensional affine-symplectic space over a finite field $\mathbb{F}_q$, and $2r \le m_0 \le i \le m \le \nu + r$. Fix an (m_0, r)-flat $x_0 + W_0$ of $ASG(2\nu, \mathbb{F}_q)$,* *and an (m,r)-flat $x+W$ of $ASG(2\nu,\mathbb{F}_q)$ such that $x_0+W_0\subset x+W$. Then the number of (i,r)-flat x_i+A of $ASG(2\nu,\mathbb{F}_q)$, where $x_0+W_0\subset x_i+A\subset x+W$, is $N(i-m_0,0;m-m_0,0;2(\nu+r-m_0))$.*

Proof Since the affine-symplectic group $ASp_{2\nu}(\mathbb{F}_q)$ acts transitively on each set of the same type flats, we may assume that W has the matrix representation of the form

$$W=\begin{matrix} & r & m_0-2r & \nu+r-m_0 & r & m_0-2r & \nu+r-m_0 & \\ & \left(\begin{matrix} I \\ 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0\\0\\I\\0\end{matrix} & \begin{matrix} 0\\0\\0\\W_1\end{matrix} & \begin{matrix} 0\\I\\0\\0\end{matrix} & \begin{matrix} 0\\0\\0\\0\end{matrix} & \left.\begin{matrix} 0\\0\\0\\W_2\end{matrix}\right) & \begin{matrix} r\\ r\\ m_0-2r\\ m-m_0\end{matrix}\end{matrix}$$

where (W_1,W_2) is an subspace of type $(m-m_0,0)$ in $\mathbb{F}_q^{2(\nu+r-m_0)}$. By (2), the number of subspaces A of type (i,r) is $N(i-m_0,0;m-m_0,0;2(\nu+r-m_0))$. It follows that the number of (i,r)-flats x_i+A is $N(i-m_0,0;m-m_0,0;2(\nu+r-m_0))$. □

Proposition 2.2 (see [5,15]) *Let $F_1=x_1+V_1$ and $F_2=x_2+V_2$ be any two flats of $ASG(2\nu,\mathbb{F}_q)$, where V_1 and V_2 are two vector subspaces of $F_q^{(2\nu)}$, and $x_1\in F_1, x_2\in F_2$. Then $F_1\cap F_2\neq\emptyset$ if and only if $x_2-x_1\in V_1+V_2$. In particular, if $F_1\cap F_2\neq\emptyset$, then $F_1\cap F_2=x+V_1\cap V_2$, where $x\in F_1\cap F_2$.*

3 Construction I

Definition 3.1 For $2r\le d_0<d<k\le\nu+r$, assume that y_0+P_0 is a fixed (d_0,r)-flat of $ASG(2\nu,\mathbb{F}_q)$. Let M be a binary matrix whose columns (rows) indexed by all (k,r)- flats containing y_0+P_0((d,r)-flats containing y_0+P_0) in $ASG(2\nu,\mathbb{F}_q)$ such that $M(x+A,y+B)=1$ if $x+A\subseteq y+B$ and 0 otherwise. This matrix is denoted by $M_1(\nu,d,k)$.

Theorem 3.1 *Suppose $2r\le d_0<d<k\le\nu+r$ and set $b=\frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1}$. Then $M_1(\nu,d,k)$ is $s^e-disjunct$ for $1\le s\le b$ and*

$$\begin{aligned} e&=q^{k-d}N(d-d_0-1,0;k-d_0-1,0;2(\nu+r-d_0))\\ &\quad-(s-1)q^{k-d-1}N(d-d_0-1,0;k-d_0-2,0;2(\nu+r-d_0)).\end{aligned}$$

Proof Let $C,C_1,\ldots,C_s$ be $s+1$ distinct columns of $M_1(\nu,d,k)$. By Proposition 2.2, to obtain the maximum numbers of (d,r)-flats which contain y_0+P_0 in

$$C\cap\bigcup_{i=1}^{s}C_i=\bigcup_{i=1}^{s}(C\cap C_i),$$

we may assume that each $C\cap C_i(1\le i\le s)$ is a $(k-1,r)$-flat.

Then each $C\cap C_i$ covers $N(d-d_0,0;k-d_0-1,0;2(\nu+r-d_0))(d,r)$-flats containing y_0+P_0 from Lemma 2.1. However, the coverage of each pair of C_i and C_j overlaps at a $(k-2,r)$-flat containing y_0+P_0, where $1\le i,j\le s$. Therefore, from Lemma 2.1 only C_1 covers the full $N(d-d_0,0;k-d_0-1,0;2(\nu+r-d_0))(d,r)$-flats containing y_0+P_0, while each of $C_2,\ldots,C_s$ can cover a maximum of $N(d-d_0,0;k-d_0-1,0;2(\nu+r-d_0))-N(d-d_0,0;k-d_0-2,0;2(\nu+r-d_0))(d,r)$-flats not covered by C_1. By (2), the

Table 1 Parameters of construction I

q	2	9	16	25
d_0	4	5	7	10
d	7	8	9	12
k	14	15	16	17
r	1	2	3	4
ν	100	110	120	130
b	8.0472	729.0002	256	625
s	2	8	41	62
e	21.3333	1.6608×10^3	6.4852×10^{16}	8.9645×10^{13}

(d, r)-flats of C not covered by $C_1, C_2, \ldots, C_s$ is at least

$$\begin{aligned} e &= N(d - d_0, 0; k - d_0, 0; 2(\nu + r - d_0)) \\ &\quad - N(d - d_0, 0; k - d_0 - 1, 0; 2(\nu + r - d_0)) \\ &\quad - (s - 1)(N(d - d_0, 0; k - d_0 - 1, 0; 2(\nu + r - d_0)) \\ &\quad - N(d - d_0, 0; k - d_0 - 2, 0; 2(\nu + r - d_0))) \\ &= q^{k-d} N(d - d_0 - 1, 0; k - d_0 - 1, 0; 2(\nu + r - d_0)) \\ &\quad - (s - 1)q^{k-d-1} N(d - d_0 - 1, 0; k - d_0 - 2, 0; 2(\nu + r - d_0)). \end{aligned}$$

Note that $\dfrac{N(d - d_0 - 1, 0; k - d_0 - 1, 0; 2(\nu + r - d_0))}{N(d - d_0 - 1, 0; k - d_0 - 2, 0; 2(\nu + r - d_0))} = \dfrac{q^{k-d_0-1} - 1}{q^{k-d} - 1}$, by (2). Since $e > 0$,

$$s < \frac{q(q^{k-d_0-1} - 1)}{q^{k-d} - 1} + 1.$$

Set

$$b = \frac{q(q^{k-d_0-1} - 1)}{q^{k-d} - 1}.$$

Then $1 \le s \le b$. □

Remark Take $d = d_0 + 1$ in Theorem 3.1, we have $b = q$ and $s \le q$. So, for a large scope of s, it is enough to take $b = q$. We give the Table 1 to see the relations of parameters of construction I.

Corollary 3.2 *Suppose that* $2r \le d_0 < d < k \le \nu + r$ *and* $1 \le s \le \min\{b, q + 1\}$. *Then* $M_1(\nu, d, k)$ *is not* s^{e+1}*-disjunct, where b and e are as in* Theorem 3.1.

Proof Let C be a (k, r)-flat containing $y_0 + P_0$, and E be a fixed $(k - 2, r)$-flat containing $y_0 + P_0$ and contained in C. By Lemma 2.1, we obtain the number of $(k - 1, r)$-flats containing E and contained in C is

$$N(1, 0; 2, 0; 2(\nu + r - k + 2)) = q + 1.$$

For $1 \le s \le \min\{b, q + 1\}$, we choose s distinct $(k - 1, r)$-flats containing E and contained in C, denote these flats by $x_i + Q_i (1 \le i \le s)$. For each $x_i + Q_i$, we choose a (k, r)-flat C_i

such that $C \cap C_i = x_i + Q_i$, where $1 \leq i \leq s$. Hence each pair of C_i and C_j overlaps at the same $(k-2, r)$-flat E, where $1 \leq i, j \leq s$. By Theorem 3.1, it follows that the corollary hold. □

Corollary 3.3 *Suppose that* $k = d + 2 = d_0 + 3$ *and* $1 \leq s \leq q$. *Then* $M_1(\nu, d, k)$ *is* s^e*-disjunct, but it is not* s^{e+1}*-disjunct, where* $e = q^2 - q(s-1)$.

Proof Setting $k = d + 2 = d_0 + 3$ in the e formula of Theorem 3.1, we obtain

$$e = q^2 - q(s-1).$$

The second statement follows directly from Corollary 3.2. □

The number of (m, r)-flats containing a given (m_0, r)-flat is equal to $N'(m_0, r; m, r; 2\nu)$. The following theorem tells us how to choose m so that the test to item ratio is minimized.

Theorem 3.4 *For* $2r \leq m_0 < m \leq \nu + r$, *the sequence* $N'(m_0, r; m, r; 2\nu)$ *is unimodal and gets its peak at* $m = \lfloor \frac{2\nu+2r+m_0}{3} \rfloor$ *or* $m = \lfloor \frac{2\nu+2r+m_0}{3} \rfloor + 1$.

Proof For $2r \leq m_0 \leq m_1 < m_2 \leq \nu + r$, by (3), we have

$$\begin{aligned} \frac{N'(m_0, r; m_1, r; 2\nu)}{N'(m_0, r; m_2, r; 2\nu)} &= \frac{\prod_{i=m_1-m_0+1}^{m_2-m_0}(q^i - 1)}{\prod_{i=\nu+r-m_2+1}^{\nu+r-m_1}(q^{2i} - 1)} \\ &= \frac{\prod_{i=0}^{m_2-m_1-1}(q^{m_1-m_0+1+i} - 1)}{\prod_{i=0}^{m_2-m_1-1}(q^{2(\nu+r-m_2+1+i)} - 1)} \\ &= \prod_{i=0}^{m_2-m_1-1} \frac{q^{m_1-m_0+1+i} - 1}{q^{2(\nu+r-m_2+1+i)} - 1}. \end{aligned} \tag{4}$$

If $\lfloor \dfrac{2\nu+2r+m_0}{3} \rfloor + 1 \leq m_1 < m_2 \leq \nu + r$, then $\frac{2\nu+2r+m_0}{3} < m_1$. It implies that

$$2m_1 + m_2 > 3m_1 > 2\nu + 2r + m_0. \tag{5}$$

Since $i \leq m_2 - m_1 - 1$, by (5) we have

$$m_1 + 2m_2 > 2\nu + 2r + m_0 + 1 + (m_2 - m_1 - 1) \geq 2\nu + 2r + m_0 + 1 + i.$$

So

$$m_1 - m_0 + 1 + i > 2(\nu + r - m_2 + 1 + i).$$

It follows that

$$q^{2(\nu+r-m_2+1+i)} - 1 < q^{m_1-m_0+1+i} - 1.$$

So

$$\frac{q^{m_1-m_0+1+i} - 1}{q^{2(\nu+r-m_2+1+i)} - 1} > 1.$$

From (4) we have

$$N'(m_0, r; m_2, r; 2\nu) < N'(m_0, r; m_1, r; 2\nu).$$

If $2r \le m_0 \le m_1 < m_2 \le \lfloor \frac{2\nu + 2r + m_0}{3} \rfloor$, then $m_2 \le \frac{2\nu+2r+m_0}{3}$. Thus

$$m_1 + 2m_2 < 3m_2 \le 2\nu + 2r + m_0 < 2\nu + 2r + m_0 + 1 + i.$$

It follows that

$$m_1 - m_0 + 1 + i < 2\nu + 2r - 2m_2 + 2 + 2i = 2(\nu + r - m_2 + 1 + i).$$

So

$$q^{m_1 - m_0 + 1 + i} - 1 < q^{2(\nu + r - m_2 + 1 + i)} - 1.$$

It follows that

$$\frac{q^{m_1 - m_0 + 1 + i} - 1}{q^{2(\nu + r - m_2 + 1 + i)} - 1} < 1.$$

From (4) we have

$$N'(m_0, r; m_2, r; 2\nu) > N'(m_0, r; m_1, r; 2\nu).$$

□

4 Discussions of test efficiency for construction I

Identifying most positive items with least tests is one of our goals. Therefore, discussing how to make the ratio t/n smaller is significative. The number of (d, r)-flats containing a given (d_0, r)-flat is equal to $N'(d_0, r; d, r; 2\nu)$. In our matrix,

$$t/n = \frac{N'(d_0, r; d, r; 2\nu)}{N'(d_0, r; k, r; 2\nu)} = \frac{\prod_{i=d-d_0+1}^{k-d_0}(q^i - 1)}{\prod_{i=\nu+r-k+1}^{\nu+r-d}(q^{2i} - 1)}.$$

We first will explain several facts on the ratio:

(1) Parameter $d_0(\nu, r)$ only appears in the numerator(denominator). It is easy to show that the larger the d_0 , ν and r are, the smaller the ratio is.
(2) The smaller the k is, the smaller(larger) the numerator(denominator) is, then the smaller the ratio is.

In [3], D'yachkov et al. constructed a pooling design with subspaces of $GF(q)$, where q is a prime power, each of the columns(rows) is labeled by an $k(d)$-dimensional space, $m_{ij} = 1$ if and only if the label of row i is contained in the label of column j. In order to compare with t/n, we should take the dimension of the space of $GF(q)$ to be $2(\nu + r - d_0)$ by the construction of (I). Assume that the efficiency ratio of [3] is t_1/n_1. Then

$$t_1/n_1 = \frac{\begin{bmatrix} 2(\nu + r - d_0) \\ d \end{bmatrix}_q}{\begin{bmatrix} 2(\nu + r - d_0) \\ k \end{bmatrix}_q} = \frac{\prod_{i=d+1}^{k}(q^i - 1)}{\prod_{i=2(\nu+r-d_0)-k+1}^{2(\nu+r-d_0)-d}(q^i - 1)}$$

Theorem 4.1 *If* $2d_0 > k - 1$, *then* $t/n < q^{(d-k)d_0} t_1/n_1$, *where* $2r \le d_0 < d < k \le \nu + r$.

Proof

$$\begin{aligned}\frac{t}{n}\Big/\frac{t_1}{n_1} &= \frac{\prod_{i=d-d_0+1}^{k-d_0}(q^i-1)}{\prod_{i=\nu+r-k+1}^{\nu+r-d}(q^{2i}-1)}\Big/\frac{\prod_{i=d+1}^{k}(q^i-1)}{\prod_{i=2(\nu+r-d_0)-k+1}^{2(\nu+r-d_0)-d}(q^i-1)}\\ &= \frac{\prod_{i=0}^{k-d-1}(q^{d-d_0+1+i}-1)}{\prod_{i=0}^{k-d-1}(q^{2(\nu+r-k+1+i)}-1)}\Big/\frac{\prod_{i=0}^{k-d-1}(q^{d+1+i}-1)}{\prod_{i=0}^{k-d-1}(q^{2(\nu+r-d_0)-k+1+i}-1)}\\ &= \prod_{i=0}^{k-d-1}\frac{q^{d-d_0+1+i}-1}{q^{d+1+i}-1}\prod_{i=0}^{k-d-1}\frac{q^{2(\nu+r-d_0)-k+1+i}-1}{q^{2(\nu+r-k+1+i)}-1}\\ &< \prod_{i=0}^{k-d-1}\frac{q^{d-d_0+1+i}}{q^{d+1+i}}\prod_{i=0}^{k-d-1}\frac{q^{2(\nu+r-d_0)-k+1+i}-1}{q^{2(\nu+r-k+1+i)}-1}\\ &= q^{(d-k)d_0}\prod_{i=0}^{k-d-1}\frac{q^{2(\nu+r-d_0)-k+1+i}-1}{q^{2(\nu+r-k+1+i)}-1}.\end{aligned}$$

Considering that $2d_0 > k-1$, so $\dfrac{q^{2(\nu+r-d_0)-k+1+i}-1}{q^{2(\nu+r-k+1+i)}-1} < 1$. Therefore $t/n < q^{(d-k)d_0}$ t_1/n_1, where $2r \le d_0 < d < k \le \nu + r$. □

5 Construction II

Definition 5.1 For $2 \le 2r \le d < k \le \nu+r$, let M be a binary matrix whose columns (rows) indexed by all (k, r)-flats ((d, r)-flats) in $ASG(2\nu, \mathbb{F}_q)$ such that $M(x + A, y + B) = 1$ if $x + A \subseteq y + B$ and 0 otherwise. This matrix is denoted by $M_2(\nu, d, k)$.

Theorem 5.1 *Suppose* $4 \le 2r + 2 \le d < k - 1 \le \nu + r - 1$. *If* $1 \le s \le q^{2r+1}$, *then* $M_2(\nu, d, k)$ *is* s^e*-disjunct, where* $e = q^{(k-d-1)(d+1)+2r+1}$.

Proof Let $C, C_1, \ldots, C_s$ be $s + 1$ distinct columns of $M_2(\nu, d, k)$. By Proposition 2.2, to obtain the maximum number of (d, r)-flats in

$$C \cap \bigcup_{i=1}^{s} C_i = \bigcup_{i=1}^{s}(C \cap C_i),$$

we may assume that each $C \cap C_i$ is a $(k-1, r)$-flat, where $1 \le i \le s$. By (2), the number of the (d, r)-flats of C not covered by $C_1, C_2, \ldots, C_s$ is at least

$$\begin{aligned}&q^{k-d}N(d, r; k, r; 2\nu) - sq^{k-1-d}N(d, r; k-1, r; 2\nu)\\ &\quad= q^{(2r+1)(k-d)}\frac{\prod_{i=k-d+1}^{k-2r}(q^i-1)}{\prod_{i=1}^{d-2r}(q^i-1)} - sq^{(2r+1)(k-d-1)}\frac{\prod_{i=k-d}^{k-2r-1}(q^i-1)}{\prod_{i=1}^{d-2r}(q^i-1)}\\ &\quad= q^{(2r+1)(k-d-1)}\frac{\prod_{i=k-d+1}^{k-2r-1}(q^i-1)}{\prod_{i=1}^{d-2r}(q^i-1)}(q^{k+1} - q^{2r+1} - s(q^{k-d}-1)).\end{aligned}$$

Since $2r + 2 \leq d < k - 1$, we obtain

$$\frac{\prod_{i=k-d+1}^{k-2r-1}(q^i - 1)}{\prod_{i=1}^{d-2r}(q^i - 1)} = \frac{\prod_{i=0}^{d-2r-2}(q^{i+k-d+1} - 1)}{\prod_{i=0}^{d-2r-2}(q^{i+1}-1)} \frac{1}{q^{d-2r} - 1}$$
$$= \prod_{i=0}^{d-2r-2} \frac{q^{i+k-d+1} - 1}{q^{i+1} - 1} \frac{1}{q^{d-2r} - 1}$$
$$= \prod_{i=0}^{d-2r-2} q^{k-d} \frac{q^{i+1} - \frac{1}{q^{k-d}}}{q^{i+1} - 1} \frac{1}{q^{d-2r} - 1}$$
$$> q^{(d-2r-1)(k-d)-(d-2r)}.$$

Since $1 \leq s \leq q^{2r+1}$, and $2r + 2 \leq d$, we obtain

$$q^{k+1} - q^{2r+1} - s(q^{k-d} - 1) \geq q^{k+1} - q^{2r+1} - q^{2r+1}(q^{k-d} - 1) = q^{k-d+2r+1}(q^{d-2r} - 1) \geq q^{k-d+2r+1}.$$

Hence $e = q^{(k-d-1)(d+1)+2r+1}$. □

Theorem 5.2 *Suppose $2 \leq 2r \leq d < \nu + r$. Let $p = \frac{q^{d+2} - q^{2r+1}}{q - 1} - 1$. If $1 \leq s \leq p$, then $M_2(\nu, d, d + 1)$ is fully s^e-disjunct, where $e = p - s + 1$.*

Proof By (2), we have $qN(d, r; d + 1, r; 2\nu) = p + 1$. It follows that we can pick $s + 1$ distinct $(d+1, r)$-flats $C, C_1, \ldots, C_s$ such that $C \cap C_i$ and $C \cap C_j$ are two distinct (d, r)-flats, where $1 \leq i, j \leq s$. By the principle of inclusion and exclusion, the number of (d, r)-flats in C but not in each C_i is $p - s + 1$, where $1 \leq i \leq s$. It follows that $e \leq p - s + 1$.

On the other hand, similar to the proof of Theorem 5.4 we obtain

$$e \geq qN(d, r; d + 1, r; 2\nu) - s = p - s + 1.$$

Hence $e = p - s + 1$. □

The following theorem tells us how to choose m so that the test to item ratio is minimized.

Theorem 5.3 *For $2r \leq m \leq \nu + r$, the sequence $q^{2\nu - m}N(m, r; 2\nu)$ is unimodal and gets its peak at $m = \lfloor \frac{2\nu+2r}{3} \rfloor$ or $m = \lfloor \frac{2\nu+2r}{3} \rfloor + 1$.*

Proof For $2r \leq m_1 < m_2 \leq \nu + r$. By (1), we have

$$\frac{q^{2\nu-m_2}N(m_2, r; 2\nu)}{q^{2\nu-m_1}N(m_1, r; 2\nu)} = \frac{\prod_{i=\nu+r-m_2+1}^{\nu+r-m_1}(q^{2i} - 1)}{q^{m_2-m_1}\prod_{i=m_1-2r+1}^{m_2-2r}(q^{2r+i} - q^{2r})}$$
$$= \frac{\prod_{i=0}^{m_2-m_1-1}(q^{2(\nu+r-m_2+1+i)} - 1)}{\prod_{i=0}^{m_2-m_1-1}(q^{m_1+2+i} - q^{2r+1})}$$
$$= \prod_{i=0}^{m_2-m_1-1} \frac{q^{2(\nu+r-m_2+1+i)} - 1}{q^{m_1+2+i} - q^{2r+1}}. \tag{6}$$

If $\lfloor \frac{2\nu + 2r}{3} \rfloor + 1 \leq m_1 < m_2 \leq \nu + r$, then $\frac{2\nu+2r}{3} < m_1$. It implies that

$$2m_1 + m_2 > 3m_1 > 2\nu + 2r. \tag{7}$$

Since $i \leq m_2 - m_1 - 1$, by (5) we have

$$m_1 + 2m_2 > 2\nu + 2r + 1 + (m_2 - m_1 - 1) \geq 2\nu + 2r + 1 + i.$$

So

$$m_1 + 1 + i > 2(\nu + r - m_2 + 1 + i).$$

It follows that

$$m_1 + 1 + i - (2r + 1) > 2(\nu + r - m_2 + 1 + i) - (2r + 1).$$

So

$$q^{2(\nu+r-m_2+1+i)-(2r+1)} < q^{m_1+1+i-(2r+1)},$$

and hence

$$q^{2(\nu+r-m_2+1+i)-(2r+1)} - \frac{1}{q^{2r+1}} < q^{m_1+1+i-(2r+1)} + [(q-1)q^{m_1+1+i-(2r+1)} - 1]$$
$$= q^{m_1+2+i-(2r+1)} - 1.$$

It follows that

$$\frac{q^{2(\nu+r-m_2+1+i)-(2r+1)} - \frac{1}{q^{2r+1}}}{q^{m_1+2+i-(2r+1)} - 1} < 1.$$

So

$$\frac{q^{2(\nu+r-m_2+1+i)} - 1}{q^{m_1+2+i} - q^{2r+1}} < 1.$$

From (6) we have

$$q^{2\nu-m_2} N(m_2, r; 2\nu) < q^{2\nu-m_1} N(m_1, r; 2\nu).$$

If $2r \leq m_1 < m_2 \leq \lfloor \frac{2\nu+2r}{3} \rfloor$, then $m_2 \leq \frac{2\nu+2r}{3}$. Thus

$$m_1 + 2m_2 < 3m_2 \leq 2\nu + 2r < 2\nu + 2r + 1 + i.$$

It follows that

$$m_1 + 2 + i \leq 2\nu + 2r - 2m_2 + 2 + 2i = 2(\nu + r - m_2 + 1 + i).$$

So

$$q^{m_1+2+i} - q^{2r+1} \leq q^{2(\nu+r-m_2+1+i)} - q^{2r+1} < q^{2(\nu+r-m_2+1+i)} - 1.$$

It follows that

$$\frac{q^{2(\nu+r-m_2+1+i)} - 1}{q^{m_1+2+i} - q^{2r+1}} > 1.$$

From (6) we have

$$q^{2\nu-m_2} N(m_2, r; 2\nu) > q^{2\nu-m_1} N(m_1, r; 2\nu).$$

□

Let $\frac{t}{n}$ be the efficiency ratio of $M_2(\nu, 2r, 2r+1)$ and let t_1/n_1 the efficiency ratio of [3]. Then we have

Theorem 5.4 *If* $d = 2r, k = 2r+1$, *then* $t/n < q^{1-2r} t_1/n_1$.

Proof If $d = 2r, k = 2r+1$, then

$$\frac{t}{n} = \frac{q^{2\nu-d} N(d, r; 2\nu)}{q^{2\nu-k} N(k, r; 2\nu)} = \frac{q N(2r, r; 2\nu)}{N(2r+1, r; 2\nu)} = \frac{q^2 - q}{q^{2\nu-2r} - 1},$$

and

$$\frac{t_1}{n_1} = \frac{\begin{bmatrix} 2\nu \\ d \end{bmatrix}_q}{\begin{bmatrix} 2\nu \\ k \end{bmatrix}_q} = \frac{\begin{bmatrix} 2\nu \\ 2r \end{bmatrix}_q}{\begin{bmatrix} 2\nu \\ 2r+1 \end{bmatrix}_q} = \frac{q^{2r+1} - 1}{q^{2\nu-2r} - 1}.$$

Therefore

$$\frac{t}{n} / \frac{t_1}{n_1} = \frac{q^2 - q}{q^{2r+1} - 1} < \frac{q^2 - 1}{q^{2r+1} - 1} < \frac{q^2}{q^{2r+1}} = q^{1-2r}$$

It follows that $t/n < q^{1-2r} t_1/n_1$. □

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